Minimal-order observer and output-feedback stabilization control design of stochastic nonlinear systems

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Abstract A minimal-order observer and output-feedback stabilization control are given for single-input multi-output stochastic nonlinear systems with unobservable states, unmodelled dynamics and stochastic disturbances. Based on the observer designed, the estimates of all observable states of the system are given, and the convergence of the estimation errors are analyzed. In addition, by using the integrator backstepping approach, an output-feedback stabilization control is constructively designed, and sufficient conditions are obtained under which the closed-loop system is asymptotically stable in the large or bounded in probability, respectively.

Keywords: **nonlinear systems, minimal-order, integrator backstepping approach, stabilization control.**

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The design of global stabilization controls for stochastic nonlinear systems has been intensively investigated[1−8], which is based on recursive applications of cascade designs, such as the well-known integrator backstepping method. Khas'minskii[9] presented the basic stability theory of stochastic control systems in his classical book, and introduced two important stability concepts: bounded in probability and asymptotically stable in the large, which have now been applied widely. It is well known that dealing with the second derivative terms is the key to stochastic control design. The existing methods deal with the second derivative terms by increasing the power of the variables in control laws[1−3] or enlarging the power of the feedback capacity[5−7]. For instance, by adopting quartic Lyapunov function to increase the power of the variables in control laws, refs. [1—3] presented asymptotical stabilization controls in the large under the assumption (A): "the nonlinearities and disturbance equal zero at the origin." Besides, refs. [5—7] studied optimal control design by using weighted quadratic Lyapunov function and adjusting the feedback capacity via regulating the weighted functions under risk-sensitive index, and pointed out that if the control goal is to stabilize the closed-loop system, then the assumption (A) mentioned above is not a necessary condition, although it seems unavoidable for globally asymptotical stabilization control.

The controls in refs. [3, 7] were based on full state feedbacks, and those in refs. [1, 5, 6, 8] were based on output feedback and full-order state observers. When there is no unmodelled dynamics and stochastic disturbance, by using the full-order observers given in refs. [1, 5, 6, 8], the state estimation error can be made converging to zero asymptotically, and the convergence and convergent rate depend only on the initial value of the estimation error, instead of the output or state processes. Recently, in ref. [10] Jiang gave a reduced-order observer with a special structure for deterministic systems. Unlike refs. [1, 5, 8], an extra nonlinear term depending on the output $y = x_1$ arises in the dynamical equation of the state estimation error. Generally speaking, this nonlinear term is not zero, even when there is no unmodelled dynamics and stochastic disturbance. This may affect the asymptotical convergence of the state estimation error. In addition, the reduced-order observer and design idea of ref. [10] are not adequate to the multi-output case (for example, $y = (x_1, \cdots, x_r)^\tau, \quad r > 1$).

The purpose of this paper is to study the design problem of output-feedback stabilization control for a class of single-input multi-output (SIMO) stochastic nonlinear systems. By introducing a minimal-order observer, an output stabilization control is constructively designed so that the closed-loop system is asymptotically stable in the large when the nonlinearities and stochastic disturbance vector fields equal zero at the equilibrium point, and is bounded in probability when the stochastic disturbance vector fields do not equal zero at the equilibrium point. The minimal-order observer introduced not only preserves the advantages of full-order observer[1,5,6,8], but also avoids the above-mentioned extra nonlinear term in the dynamical equation of the state estimation error[10], and at the same time, is adequate to the control design of SIMO systems.

1 Notations and preliminary results

The following notations will be used throughout this paper. For a given vector or matrix X, X^τ denotes its transpose; $\|X\|$ denotes the Euclidean norm in vector case or the corresponding induced norm in matrix case; tr(X) denotes its trace when X is square, i.e. the sum of all elements on the main diagonal line. I denotes the identity matrix (the dimension will be determined in accordance with the context). For a given vector $x = (x_1, \cdots, x_n)^\tau$, $x_{[i]}$ denotes $(x_1, \cdots, x_i)^\tau$; $x_{[i,j]}$ denotes $(x_i, \cdots, x_j)^\tau$; $\hat{x}$ denotes its estimate associated with an observer, $\tilde{x}$ denotes estimation error, i.e. $\tilde{x} = x - \hat{x}$. For a given scalar number x, $|x|$ denotes its absolute value.

For simplicity of expression, we will drop the arguments of functions when no confusion is caused.

Definition 1[4]**.** A function $\delta(\cdot)$: $\mathbb{R}_+ \to \mathbb{R}_+$ is called $\mathcal{K}_\infty$ if it is continuous, strictly increasing, and $\delta(0) = 0$, $\lim_{\mu\to\infty} \delta(\mu) = \infty$.

For stochastic nonlinear time-varying systems in form:

$$dx = f(t, x)dt + g(t, x)udt + h(t, x)dw,$$

where w is standard Brownian motion with appropriate dimension defined on probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with Ω a sample space, $\mathcal{F}$ a σ-algebra, $\mathcal{P}$ a probability measure, we define the differential operator $\mathcal{L}$ as the following:

$$\mathcal{L}V(t,x) = \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(t,\, x) + \frac{\partial V(t,x)}{\partial x} g(t,\, x)u + \frac{1}{2}\mathrm{tr}\left\{ \frac{\partial^2 V(t,x)}{\partial x^2} h(t,\, x) h^{\tau}(t,\, x) \right\}.$$

Here $V(t,\, x)$ is a function once continuously differentiable with respect to t and twice continuously differentiable with respect to x.

Recall two stability notions for the following free-control system:

$$dx = f(t,\, x)dt + h(t,\, x)dw. \tag{1}$$

Definition 2[9]**.** Consider system (1) with $f(0) = 0$ and $h(0) = 0$. Let $\{x(t),\, t \geqslant 0\}$ be the solution precess of system (1) with initial value $x(0)$. The zero solution $x(t) = 0$ or system (1) is said to be asymptotically stable in the large if for any $\varepsilon > 0$,

$$\lim_{x(0)\to 0} P\left\{ \sup_{t\geqslant 0} \|x(t)\| \geqslant \varepsilon \right\} = 0,$$

and for any initial condition $x(0)$,

$$P\left\{ \lim_{t\to\infty} x(t) = 0 \right\} = 1.$$

Definition 3[9]**.** System (1) is said to be bounded in probability, if all of its solution processes $\{x(t),\, t \geqslant 0\}$ satisfy

$$\lim_{c\to\infty} \sup_{0\leqslant t<\infty} P\left\{ \|x(t)\| > c \right\} = 0.$$

Based on these two concepts, we have the following basic theorem, which will play an important role in our control design below.

Theorem 1. Consider the stochastic nonlinear system (1). If there exists a function $V(t,\, x)$ once continuously differential with respect to t and twice continuously differential with respect to x, and satisfying

$$W_1(x) \leqslant V(t,\, x) \leqslant W_2(x), \quad \mathcal{L}V(t,\, x) \leqslant -c_1 V(t,\, x) + c_2,$$

where $W_1(x)$ and $W_2(x)$ are positive definite and radially unbounded functions, $c_1 > 0$ and $c_2 \geqslant 0$ are constants, then (a) system (1) has a unique solution almost surely, (b) system (1) is bounded in probability, (c) in addition, if $f(t,\, 0) = 0$, $h(t,\, 0) = 0$, $\forall t$, and there exists a positive definite and radially unbounded function $W(x)$ such that

$$\mathcal{L}V(t,\, x) \leqslant -W(x),$$

then system (1) is asymptotically stable in the large.

Proof. By Theorem 4.1 of Chapter 3 and Theorem 4.4 of Chapter 5 of ref. [9], Theorem 2 of Chapter 3 and Section 13 of ref. [11], we can show the theorem in a similar way to the proof of Theorem 2.5 of ref. [7].

2 Problem formulation

2.1 System model

Consider the following stochastic nonlinear system

$$\begin{aligned} d\chi &= \sigma_0(t,\chi,x)dt + \sigma_1(t,\chi,x)dw, \\ dx_1 &= x_2dt + f_1(x_{[1]})dt + \psi_1(t,\,\chi,\,x)dt + \varphi_1(x_1)dw, \\ &\vdots \\ dx_r &= x_{r+1}dt + f_r(x_{[r]})dt + \psi_r(t,\,\chi,\,x)dt + \varphi_r(x_1)dw, \\ dx_{r+1} &= x_{r+2}dt + f_{r+1}(y)dt + \psi_{r+1}(t,\,\chi,\,x)dt + \varphi_{r+1}(x_1)dw, \\ &\vdots \\ dx_n &= udt + f_n(y)dt + \psi_n(t,\,\chi,\,x)dt + \varphi_n(x_1)dw, \\ y &= x_{[r]}, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}^r$ are system state vector, control input and measurable output, respectively; $\chi \in \mathbb{R}^{n_0}$ is the unobservable states of the system, its dynamical model is unknown, i.e. functions $\sigma_0(t,\chi,x)$ and $\sigma_1(t,\chi,x)$ are unknown; $w \in \mathbb{R}^m$ is an independent vector-valued standard Brownian motion defined on probability space $(\Omega,\mathcal{F},\mathcal{P})$, with Ω a sample space, $\mathcal{F}$ a σ-algebra, $\mathcal{P}$ a probability measure; $f_i(x_{[i]})$ $(i=1,\cdots,r)$ and $f_i(y)$ $(i=r+1,\cdots,n)$ are the modelled (or known) dynamics of the system; $\psi_i(t,\chi,\,x)$ $(i=1,\cdots,n)$ are the unmodelled (or unknown) dynamics of the system; $\varphi_i(x_1)$ $(i=1,\cdots,n)$ are the gain functions of the stochastic disturbances.

The main results of this paper are based on the following assumptions:

A1. For the unobservable states χ, there exists a time-varying $V_0(t,\chi)$ once continuously differential with respect to t and twice continuously differential with respect to x, and there exist positive definite and radially unbounded functions $W_{01}(\chi)$ and $W_{02}(\chi)$, a $\mathcal{K}_\infty$ function $\delta(\cdot)$, a smooth function $\delta_0(\cdot)$: $\delta_0(0)=0$, and constants $c>0$, $\gamma_0>0$, such that

$$W_{01}(\chi) \leqslant V_0(t,\,\chi) \leqslant W_{02}(\chi) \quad \text{and}$$

$$\mathcal{L}V_0(t,\chi) \leqslant -cV_0(t,\,\chi) - \gamma_0\delta\left(\|\chi\|\right) + x_1\delta_0(x_1).$$

A2. Nonlinear functions $f_i(\cdot)$ $(i=1,\cdots,n)$ and $\varphi_i(\cdot)$ $(i=1,\cdots,n)$ are known and smooth, and satisfy: $f_i(0)=0$ $(i=1,\cdots,n)$.

A3. There exist a $\mathcal{K}_\infty$ function $\pi(\cdot)$, a nonnegative and smooth function $\delta_i(\cdot)$: $\delta_i(0)=0$ $(i=1,\cdots,n)$ and constants $\gamma_i \geqslant 0$ $(i=1,\cdots,n)$ such that the unmodelled dynamics $\psi_i(t,\,\chi,\,x)$ $(i=1,\cdots,n)$ satisfy: $|\psi_i|^2 \leqslant \gamma_i\pi\left(\|\chi\|\right) + x_1^2\delta_i(x_1)$ $(i=1,\cdots,n)$.

A4. There exists a constant $\gamma>0$ such that $\delta(\mu) \geqslant \gamma\pi(\mu), \quad \forall\mu \in [0,\,\infty)$.

Remark 1. The states of system (2) are divided into two parts: one is χ, which is unobservable and unknown dynamics, and the other is x, which can be measured directly or is observable. They may depend on and affect each other. The stability of the unobservable state χ may be affected by the dynamical behavior of x, while the stability of x may be affected by the dynamical behavior of χ. Assumption A1 describes the dynamical behavior of the unobservable state χ of the system: it not only is exponentially stable in the large when $x \equiv 0$, but also has some stability margin with respect to the unmodelled dynamics. The term $x_1\delta_0(x_1)$ limits the influence of state x on the stability of the unobservable state χ. For the unobservable state χ satisfying this limitation, we can eliminate the influence of x on the stability of χ by designing control properly, and accomplish stabilization control of the unobservable states. Similarly, in order to construct an output-feedback stabilization control, Assumption A3 gives some constraints on unmodelled dynamics. Assumptions A1—A3 tell us that when there is no stochastic disturbance, the origin is the equilibrium point of the open-loop system. Assumption A4 depicts the connection between the stability margin of the unobservable state χ and the unmodelled dynamics, which, when designing a control law, ensures that the influence of the unmodelled dynamics on the stability of the unobservable state χ can be removed in virtue of the stability margin of the unobservable state.

System (2) can be rewritten into the following compact form:

$$\begin{cases} d\chi = \sigma_0(t,\chi,x)dt + \sigma_1(t,\chi,x)dw, \\ dy = \left[A_r y + B_r x_{r+1} + F_{[r]}(y) + \Psi_{[r]}(t,\,\chi,\,x)\right]dt + H_{[r]}(x_1)dw, \\ dx_{[r+1,n]} = \left[A_{n-r}x_{[r+1,n]} + B_{n-r}u + F_{[r+1,n]}(y) + \Psi_{[r+1,n]}(t,\,\chi,\,x)\right]dt \\ \qquad +H_{[r+1,n]}(x_1)dw, \end{cases} \tag{3}$$

where

$$A_r = \begin{bmatrix} 0 & I \\ \vdots & \\ 0 & 0\cdots0 \end{bmatrix}_{(r\times r)}, \quad B_r = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{(r\times 1)},$$

$$F = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \quad \Psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_n \end{bmatrix}, \quad H = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix}.$$

2.2 Control objective

The objective of this paper is to design a minimal-order observer and an output-feedback control:

$$\dot{\hat{x}} = \vartheta(\hat{x},\,y), \quad u = \mu(\hat{x},\,y), \tag{4}$$

such that the zero solution of the closed-loop systems (2)—(4) is bounded in probability and, in particular, when $\varphi_i(0) = 0\,(i = 1,\,\cdots,\,n)$, asymptotically stable in the large.

3 Output-feedback control design

In this section, a minimal-order observer is introduced first, based on which estimates of all the system states are given, and then, a constructive procedure for stabilization control design is described.

Since $x_1, x_2, \cdots, x_r$, the former r components of x of system (2), can be obtained directly from the system output, only the latter $n-r$ components $x_{r+1}, \cdots, x_n$ need to be rebuilt by an observer. By the linear system theory, when system (2) degenerates to a linear system and is without unknown disturbances, the minimal order of its state observer is "$n-r$" (e.g. ref. [12]).

3.1 Observer design

Denote $q(t) = dy - A_r y dt - F_{[r]}(y)dt$, or equivalently by (3), $q(t) = B_r x_{r+1} dt + \Psi_{[r]}(t, \chi, x)dt + H_{[r]}(x_1)dw$. If $q(t)$ is available for feedback design, then a minimal-order observer can be designed as

$$d\hat{x}_{[r+1,n]} = D\hat{x}_{[r+1,n]}dt + B_{n-r}udt + F_{[r+1,n]}(y)dt + Gq(t), \tag{5}$$

where $D = A_{n-r} - GB_rC_{n-r}^\tau$, $C_{n-r} = (1 \quad 0 \quad \cdots \quad 0)^\tau \in \mathbb{R}^{n-r}$, and $G = [g_{ij}] \in \mathbb{R}^{(n-r)\times r}$ is a parameter matrix to be determined below. But, due to the existence of the unmeasurable state x_{r+1}, unknown function $\Psi_{[r]}(t, \chi, x)$ and stochastic disturbance w, eq. (5) cannot be used for feedback design. In order to overcome this difficulty, we introduce a new vector:

$$\xi = \hat{x}_{[r+1,n]} - Gy. \tag{6}$$

This together with (5) gives

$$\dot{\xi} = D\xi + B_{n-r}u + F_{[r+1,n]}(y) - GF_{[r]}(y) + (DG - GA_r)y. \tag{7}$$

Obviously, ξ is feasible, and thus, can be used for feedback design.

From (6) and (7), we achieve a reconstruction of the unmeasurable state vector $x_{[r+1,\,n]}$: $\hat{x}_{[r+1,n]} = \xi + Gy$, with estimation error

$$\tilde{x}_{[r+1,n]} = x_{[r+1,n]} - \hat{x}_{[r+1,n]} = x_{[r+1,n]} - \xi - Gy, \tag{8}$$

which satisfies the following dynamical equation:

$$\begin{aligned} d\tilde{x}_{[r+1,n]} =& \left[A_{n-r}x_{[r+1,n]} + B_{n-r}u + F_{[r+1,n]}(y) + \Psi_{[r+1,n]}(t, \chi, x)\right] dt \\ &+ H_{[r+1,n]}(x_1)dw - \left[D\xi + B_{n-r}u + F_{[r+1,n]}(y) - GF_{[r]}(y)\right. \\ &\left.+ (DG - GA_r)y\right] dt - G\left[A_r y dt + B_r C_{n-r}^\tau x_{[r+1,n]}dt\right. \\ &\left.+ F_{[r]}(y)dt + \Psi_{[r]}(t, \chi, x)dt + H_{[r]}(x_1)dw\right] \\ =& D\tilde{x}_{[r+1,n]}dt + \overline{\Psi}(t, \chi, x)dt + \overline{H}(x_1)dw, \end{aligned} \tag{9}$$

where $\overline{\Psi}(t, \chi, x) = \Psi_{[r+1,n]}(t, \chi, x) - G\Psi_{[r]}(t, \chi, x)$, $\overline{H}(x_1) = H_{[r+1,n]}(x_1) - GH_{[r]}(x_1)$.

Denote the last column of matrix G as $(k_{r+1}, \cdots, k_n)^\tau$. Assume that the polynomial

$s^{n-r} + k_{r+1}s^{n-r-1} + \cdots + k_{n-1}s + k_n$ is Hurwitz. Then the $(n-r) \times (n-r)$ matrix

$$D = \begin{bmatrix} -k_{r+1} & \\ \vdots & I \\ -k_n & 0 \cdots 0 \end{bmatrix}$$

is strictly stable. And thus, there exists a positive definite matrix P such that

$$D^\tau P + PD = -I. \tag{10}$$

It can easily be seen that if the unmodelled dynamics and the stochastic disturbance do not exist, i.e. $\Psi(t, \chi, x) \equiv 0$ and $H(x_1) \equiv 0$, then (9) becomes

$$\dot{\tilde{x}}_{[r+1,n]} = D\tilde{x}_{[r+1,n]}. \tag{11}$$

Therefore, by the strict stability of D, the estimation error $\tilde{x}_{[r+1,n]}$ is globally asymptotically stable. This implies that the unmeasurable state vector $(x_{r+1}, \cdots, x_n)^\tau$ can be reconstructed by $(\xi + Gy)$ very well.

In this case, the overall system with the observer (7) in loop is

$$\begin{aligned} d\chi =& \sigma_0(t, \chi, x)dt + \sigma_1(t, \chi, x)dw, \\ d\tilde{x}_{[r+1,n]} =& D\tilde{x}_{[r+1,n]}dt + \overline{\Psi}(t, \chi, x)dt + \overline{H}(x_1)dw, \\ dy =& \left[A_r y + B_r(\tilde{x}_{r+1} + \xi_1) + B_r C_{n-r}^\tau Gy + F_{[r]}(y) \right. \\ & \left. + \Psi_{[r]}(t, \chi, x)\right] dt + H_{[r]}(x_1)dw, \\ d\xi =& \left[D\xi + B_{n-r}u + F_{[r+1,n]}(y) - GF_{[r]}(y) + (DG - GA_r)y\right] dt. \end{aligned} \tag{12}$$

Remark 2. Following refs. [1, 8, 10], other reduce-order (or minimal-order) observers may be obtained, but the order-reduction degree and convergent rate may not be as good as the observer (7).

For example, following refs. [1, 8], an observer can be given as:

$$\dot{\hat{x}}_{[r,n]} = A_{n-r+1}\hat{x}_{[r,n]} + K_{[r,n]}(x_r - \hat{x}_r) + B_{n-r+1}u + F_{[r,n]}(y),$$

where $\hat{x}_{[r,n]}$ denote the estimate of $x_{[r,n]}$, $K_{[r,n]} = (k_r, \cdots, k_n)^\tau$ is a design parameter vector such that the polynomial $s^{n-r+1} + k_r s^{n-r} + \cdots + k_{n-1}s + k_n$ is Hurwitz.

Let $\tilde{x}_{[r,n]} = x_{[r,n]} - \hat{x}_{[r,n]}$ be the estimation error. Then

$$d\tilde{x}_{[r,n]} = S_r\tilde{x}_{[r,n]}dt + \Psi_{[r,n]}(t, \chi, x)dt + H_{[r,n]}(x_1)dw,$$

where

$$S_r = \begin{bmatrix} -k_r & \\ \vdots & I \\ -k_n & 0 \cdots 0 \end{bmatrix}$$

is a strictly stable matrix.

Obviously, in this case, although the estimation error $\tilde{x}_{[r,n]}$, and thus, $\tilde{x}$ converges asymptotically to zero when the unmodelled dynamics and the stochastic disturbance do not exist, the observer is $n - r + 1$-order and not minimal-order.

For one more example, following ref. [10], we can obtain the estimate $\hat{x}_i = \xi_i + k_i x_r$ $(i = r+1, \cdots, n)$ of state x_i $(i = r+1, \cdots, n)$, where ξ_i $(i = r+1, \cdots, n)$ are the states of the following observer:

$$\begin{aligned} \dot{\xi}_{r+1} &= \xi_{r+2} + k_{r+2}x_r - k_{r+1}\left(\xi_{r+1} + k_{r+1}x_r\right), \\ \dot{\xi}_i &= \xi_{i+1} + k_{i+1}x_r - k_i\left(\xi_{r+1} + k_{r+1}x_r\right),\ i = r+2, \cdots, n-1, \\ \dot{\xi}_n &= u - k_n\left(\xi_{r+1} + k_{r+1}x_r\right), \end{aligned} \tag{13}$$

where design parameters k_i $(r+1 \leqslant i \leqslant n)$ are chosen such that the matrix S_{r+1} is strictly stable.

Then, the estimation error $\tilde{x}_{[r+1,n]} = (x_{r+1} - \xi_{r+1} - k_{r+1}x_r,\ \cdots,\ x_n - \xi_n - k_n x_r)^\tau$ satisfies

$$d\tilde{x}_{[r+1,n]} = S_{r+1}\tilde{x}_{[r+1,n]}dt + \overline{f}(y)dt + \overline{\psi}(t,\ \chi,\ x)dt + \overline{\varphi}(x_1)dw, \tag{14}$$

where

$$\begin{aligned} \overline{f}(y) &= (f_{r+1}(y) - k_{r+1}f_r(y),\ \cdots,\ f_n(y) - k_n f_r(y))^\tau, \\ \overline{\psi}(t,\ \chi,\ x) &= (\psi_{r+1}(t,\ \chi,\ x) - k_{r+1}\psi_r(t,\ \chi,\ x),\ \cdots,\ \psi_n(t,\ \chi,\ x) - k_n\psi_r(t,\ \chi,\ x))^\tau, \\ \overline{\varphi}(x_1) &= (\varphi_{r+1}(x_1) - k_{r+1}\varphi_r(x_1),\ \cdots,\ \varphi_n(x_1) - k_n\varphi_r(x_1))^\tau. \end{aligned}$$

By this, it can easily be seen that the observer (13) is now $n-r$-order, minimal-order, but the convergence of the estimation error is hard to analyze. For instance, when the unmodelled dynamics and the stochastic disturbance do not exist, i.e. $\Psi(t,\ \chi,\ x) \equiv 0$ and $H(x_1) \equiv 0$, the error equation (14) becomes

$$\dot{\tilde{x}}_{[r+1,n]} = S_{r+1}\tilde{x}_{[r+1,n]} + \overline{f}(y).$$

Unlike (11), an extra nonlinear term $\overline{f}(y)$ arises here. Due to this unexpected term, the estimation error $\tilde{x}_{[r+1,n]}$ may not be convergent to zero, in general. Besides, the observer (13) is only applicable to the single-output systems such as $y = x_1$, and not applicable to the multi-output systems such as $y = x_{[r]}$ $(r > 1)$ (This will be explained further below).

3.2 Control design

We are now in a position to construct a control $u(y, \xi)$ for the overall system (12) to ensure the closed-loop system is bounded in probability and asymptotically stable in the large when the nonlinearities and stochastic disturbance vector field equal zero at the equilibrium point of the open-loop system.

First, introduce a new state transformation as follows:

$$\begin{cases} z_i = x_i - \alpha_{i-1}(x_{[i-1]}),\ \ i = 1, 2, \cdots, r, \\ z_i = \xi_{i-r} - \alpha_{i-1}(y, \xi_{[i-r-1]}),\ \ i = r+1, \cdots n, \end{cases} \tag{15}$$

and set $\alpha_0(x_{[0]}) \equiv 0$, $z_{n+1} \equiv 0$. Here, α_i $(i = 1, \cdots, n-1)$, called the virtual controls, are some smooth functions to be determined later; $\alpha_n = u(y,\ \xi)$ is the actual control to be specified later. Besides, α_i $(i = 1,\ \cdots,\ n)$ are asked to preserve the equilibrium at the origin of the nonlinear system, that is, $\alpha_1(0) = \cdots = \alpha_r(0) = \alpha_{r+1}(0,0) = \cdots = \alpha_n(0,0) = 0$.

Under the new variable vector z, system (12) becomes

$$
\begin{aligned}
d\chi &= \sigma_0(t,\,\chi,\,x)dt + \sigma_1(t,\,\chi,\,x)dw,\\
d\tilde{x}_{[r+1,n]} &= D\tilde{x}_{[r+1,n]}dt + \overline{\Psi}(t,\,\chi,\,x)dt + \overline{H}(x_1)dw,\\
dz_i &= (z_{i+1}+\alpha_i)dt + \Omega_i(x_{[i]})dt + \Theta_i(t,\,\chi,\,x)dt + \Phi_i(x_1,\,x_{[i-1]})dw,\\
&\qquad\qquad i = 1,\cdots,r-1,\\
dz_r &= (z_{r+1}+\alpha_r)dt + \tilde{x}_{r+1}dt + \Omega_r(y)dt + \Theta_r(t,\,\chi,\,x)dt\\
&\quad + \Phi_r(x_1,\,x_{[r-1]})dw,\\
dz_{r+i} &= (z_{r+i+1}+\alpha_{r+i})dt - \frac{\partial\alpha_{r+i-1}}{\partial x_r}\tilde{x}_{r+1}dt + \Omega_{r+i}(y,\xi_{[i]})dt\\
&\quad + \Theta_{r+i}(t,\,\chi,\,x)dt + \Phi_{r+i}(y,\xi_{[i-1]})dw,\;\; i=1,\cdots,n-r-1,\\
dz_n &= udt - \frac{\partial\alpha_{n-1}}{\partial x_r}\tilde{x}_{r+1}dt + \Omega_n(y,\xi_{[n-r]})dt + \Theta_n(t,\,\chi,\,x)dt\\
&\quad + \Phi_n(y,\,\xi_{[n-r-1]})dw,
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
\Phi_i &= \varphi_i(x_1) - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j(x_1),\quad i=1,\cdots,r;\\
\Phi_{r+i} &= -\sum_{j=1}^{r}\frac{\partial\alpha_{r+i-1}}{\partial x_j}\varphi_j(x_1),\quad i=1,\cdots,n-r,\\
\Omega_i &= f_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}[x_{j+1}+f_j] - \frac{1}{2}\sum_{j,\,k\in\{1,\cdots,i-1\}}\frac{\partial^2\alpha_{i-1}}{\partial x_j\partial x_k}\varphi_j\varphi_k^{\tau},\\
&\qquad\qquad i=1,2,\cdots,r-1,\\
\Omega_r &= f_r + \sum_{j=1}^{r}g_{1j}x_j - \sum_{j=1}^{r-1}\frac{\partial\alpha_{r-1}}{\partial x_j}[x_{j+1}+f_j] - \frac{1}{2}\sum_{j,\,k\in\{1,\cdots,r-1\}}\frac{\partial^2\alpha_{i-1}}{\partial x_j\partial x_k}\varphi_j\varphi_k^{\tau},\\
\Omega_{r+i} &= f_{r+i}(y) - k_{r+i}\xi_1 - \sum_{j=1}^{r}g_{ij}f_j + [DG-GA_r]_i y - \sum_{j=1}^{r-1}\frac{\partial\alpha_{r+i-1}}{\partial x_j}x_{j+1}\\
&\quad - \sum_{j=1}^{r}\frac{\partial\alpha_{r+i-1}}{\partial x_j}f_j(x_{[j]}) - \frac{\partial\alpha_{r+i-1}}{\partial x_r}\left(\xi_1 + \sum_{i=1}^{r}g_{1i}x_i\right) - \sum_{j=1}^{i-1}\frac{\partial\alpha_{r+i-1}}{\partial\xi_j}\\
&\quad\times\left(-k_{r+j}\xi_1 + \xi_{j+1} + f_{r+j} + [DG-GA_r]_j y + \sum_{k=1}^{r}g_{jk}f_k\right)\\
&\quad - \frac{1}{2}\sum_{j,\,k\in\{1,\cdots,r\}}\frac{\partial^2\alpha_{i-1}}{\partial x_j\partial x_k}\varphi_j\varphi_k^{\tau},\; i=1,\cdots,n-r,\\
\Theta_i &= \psi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_j, i=1,\,\cdots,\,r;\\
\Theta_{r+i} &= -\sum_{j=1}^{r}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_j,\quad i=1,\,\cdots,\,n-r.
\end{aligned}
$$

Here, $[DG - GA_r]_i$ denotes the r-dimension row vector consisting of the ith row of the matrix $DG - GA_r$.

Choose Lyapunov function $V(\cdot,\ \cdot): \mathbb{R}^{n-r} \times \mathbb{R}^n \to \mathbb{R}$:

$$V = \tilde{x}^\tau_{[r+1,n]} P \tilde{x}_{[r+1,n]} + \sum_{i=1}^n \Xi_i(z_{[i-1]}) z_i^2,$$

where $P > 0$ is the unique positive solution of (10), $\Xi_i(z_{[i-1]}) > 0$ $(i = 1, \cdots, n)$ are some smooth weighted functions to be specified later.

By Itô formula, from (10) and (16) we have

$$\begin{aligned}\mathcal{L}V = & -\|\tilde{x}_{[r+1,n]}\|^2 + 2\tilde{x}^\tau_{[r+1,\,n]} P\overline{\Psi} + \mathrm{tr}\{\overline{H}^\tau P \overline{H}\} + \sum_{i=r}^n M_i \tilde{x}_{r+1} z_i \\ & + 2\sum_{i=1}^n \Xi_i (z_{i+1} + \alpha_i + \Omega_i + \Theta_i) z_i + \sum_{i=2}^n \sum_{j=1}^{i-1} \frac{\partial \Xi_i}{\partial z_j}(z_{j+1} + \alpha_j + \Omega_j + \Theta_j) z_i^2 \\ & + \frac{1}{2}\sum_{i=1}^n \mathrm{tr}\left\{ \frac{\partial^2 (\Xi_i z_i^2)}{\partial z_{[i]}^2} [\Phi_1^\tau,\ \cdots,\ \Phi_i^\tau]^\tau [\Phi_1^\tau,\ \cdots,\ \Phi_i^\tau] \right\},\end{aligned} \tag{17}$$

where

$$\begin{aligned} & M_r = 2\Xi_r\,, \qquad M_{r+1} = \frac{\partial \Xi_{r+1}}{\partial z_r} z_{r+1} - 2\Xi_{r+1}\frac{\partial \alpha_r}{\partial x_r}\,, \\ & M_i = \frac{\partial \Xi_i}{\partial z_r} z_i - \sum_{j=r+1}^{i-1} \frac{\partial \Xi_i}{\partial z_j}\frac{\partial \alpha_{j-1}}{\partial x_r} z_i - 2\Xi_i \frac{\partial \alpha_{i-1}}{\partial x_r}\,, \quad i = r+2,\ \cdots,\ n. \end{aligned}$$

By Assumptions A3 and A4, the second term on the right-hand side of (17) satisfies

$$\begin{aligned} & 2\tilde{x}^\tau_{[r+1,\,n]} P \overline{\Psi} \leqslant \varepsilon_1 \|\tilde{x}_{[r+1,\,n]}\|^2 + \frac{\|P\|^2}{\varepsilon_1}\|\overline{\Psi}\|^2 \\ & \leqslant \varepsilon_1 \|\tilde{x}_{[r+1,\,n]}\|^2 + \frac{\|P\|^2}{\varepsilon_1}\left(\|G\|^2 + 1\right)\sum_{i=1}^n \left[\frac{\gamma_i}{\gamma}\delta\left(\|\chi\|\right) + x_1^2 \delta_i(x_1)\right], \end{aligned} \tag{18}$$

where and whereafter $\varepsilon_1, \varepsilon_0, \varepsilon_2$ and $\kappa_i, \varepsilon_{2i}, \beta_i$ $(i = 1, \cdots, n)$ are positive design parameters to be specified.

For the 4th term on the right-hand side of (17) we have

$$\begin{aligned} \sum_{i=r}^n M_i \tilde{x}_{r+1} z_i = & \frac{\varepsilon_2}{2}|\tilde{x}_{r+1}|^2 - \frac{\varepsilon_2}{2}\left|\tilde{x}_{r+1} - \frac{1}{\varepsilon_2}\sum_{i=r}^n M_i z_i\right|^2 \\ & + \frac{1}{\varepsilon_2}\sum_{i=r}^n \sum_{j=r}^{i-1} M_i z_i M_j z_j + \frac{1}{2\varepsilon_2}\sum_{i=r}^n M_i^2 z_i^2. \end{aligned} \tag{19}$$

Besides, by Assumptions A3 and A4 we have

$$\begin{aligned}2\sum_{i=1}^{n}\Xi_i\Theta_i z_i &= 2\sum_{i=1}^{r}\Xi_i z_i\psi_i - 2\sum_{i=1}^{n}\Xi_i z_i\sum_{j=1}^{\min\{i-1,\,r\}}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_j\\ &\leqslant \sum_{i=1}^{r}\left[\Xi_i^2 z_i^2 + \frac{\gamma_i}{\gamma}\delta\left(\|\chi\|\right) + z_1^2\delta_i(z_1)\right]\\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{\min\{i-1,\,r\}}\left[\left(\frac{\partial\alpha_{i-1}}{\partial x_j}\right)^2\Xi_i^2 z_i^2 + \frac{\gamma_j}{\gamma}\delta\left(\|\chi\|\right) + z_1^2\delta_j(z_1)\right] \quad (20)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=2}^{n}\sum_{j=1}^{i-1}\frac{\partial\Xi_i}{\partial z_j}\Theta_j z_i^2 &= \sum_{i=2}^{n}\left(\sum_{j=1}^{\min\{i-1,\,r\}}\frac{\partial\Xi_i}{\partial z_j}\psi_j - \sum_{j=1}^{i-1}\frac{\partial\Xi_i}{\partial z_j}\sum_{k=1}^{\min\{j-1,\,r\}}\frac{\partial\alpha_{j-1}}{\partial x_k}\psi_k\right)z_i^2\\ &\leqslant \frac{1}{2}\sum_{i=2}^{n}\sum_{j=1}^{\min\{i-1,\,r\}}\left[\left(\frac{\partial\Xi_i}{\partial z_j}\right)^2 z_i^4 + \frac{\gamma_j}{\gamma}\delta\left(\|\chi\|\right) + z_1^2\delta_j(z_1)\right]\\ &\quad + \frac{1}{2}\sum_{i=2}^{n}\sum_{j=1}^{i-1}\sum_{k=1}^{\min\{j-1,\,r\}}\left[\left(\frac{\partial\Xi_i}{\partial z_j}\right)^2\left(\frac{\partial\alpha_{j-1}}{\partial x_k}\right)^2 z_i^4\right.\\ &\quad \left. + \frac{\gamma_k}{\gamma}\delta\left(\|\chi\|\right) + z_1^2\delta_k(z_1)\right]. \quad (21)\end{aligned}$$

Denote $X = (y^\tau,\ \xi^\tau)^\tau$. Then, by the diffeomorphism (15) there exist smooth functions $\vartheta_i(\cdot)$ $(i = 1, \cdots, n)$ such that $z_{[i]} = \vartheta_i(X_{[i]})$ $(i = 1, \cdots, n)$. From the smooth properties of $\varphi_i(\cdot)$, $\alpha_i(\cdot)$ and $\vartheta_i(\cdot)$ $(i = 1, \cdots, n)$ it follows that $\overline{H}$, Φ_i can be decomposed into the following forms:

$$\begin{aligned}\overline{H}(x_1) &= \overline{H}(z_1) = \overline{H}(0) + \overline{H}_{11}(z_1)z_1,\\ \Phi_1(x_1) &= \varphi_1(z_1) = \varphi_1(0) + \overline{\Phi}_{11}(z_1)z_1,\\ \Phi_i(X_{[i-1]}) &= \overline{\Phi}_i(z_{[i-1]}) = \overline{\Phi}_i(0) + \sum_{j=1}^{i-1}\overline{\Phi}_{ij}(z_{[j]})z_j, \quad i = 2, \cdots, n,\end{aligned}$$

where $\overline{\overline{H}}(z_1)$, $\overline{\Phi}_{11}(z_1)$ and $\overline{\Phi}_{ij}(z_{[j]})$, $i = 2, \cdots, n$, $j = 1, \cdots, i-1$ are smooth functions, and $\overline{H}(0) = H_{[r+1,n]}(0) - GH_{[r]}(0)$ is available for feedback design.

Thus, we have

$$\begin{aligned}\mathrm{tr}\{\overline{H}(x_1)P\overline{H}^\tau(x_1)\} &= \mathrm{tr}\left\{\left[\overline{H}(0) + \overline{H}_{11}(z_1)z_1\right]P\left[\overline{H}(0) + \overline{H}_{11}(z_1)z_1\right]^\tau\right\}\\ &= \mathrm{tr}\left[\overline{H}(0)P\overline{H}^\tau(0)\right]\\ &\quad + \mathrm{tr}\left\{\left[2\overline{H}(0) + \overline{H}_{11}(z_1)z_1\right]P\overline{H}_{11}^\tau(z_1)\right\}z_1 \quad (22)\end{aligned}$$

and

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\mathrm{tr}\left\{\frac{\partial^2(\Xi_i z_i^2)}{\partial z_{[i]}^2}[\Phi_1^\tau,\cdots,\Phi_i^\tau]^\tau[\Phi_1^\tau,\cdots,\Phi_i^\tau]\right\}\\
=\;&\frac{1}{2}\sum_{i=1}^{n}\mathrm{tr}\left[\begin{bmatrix}\dfrac{\partial^2\Xi_i}{\partial z_{[i-1]}^2}z_i^2 & 2\dfrac{\partial\Xi_i}{\partial z_{[i-1]}}z_i\\ 2\left(\dfrac{\partial\Xi_i}{\partial z_{[i-1]}}\right)^\tau z_i & 2\Xi_i\end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_i\end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_i\end{bmatrix}^\tau\right]\\
=\;&\frac{1}{2}\sum_{i=1}^{n}\mathrm{tr}\left[\begin{bmatrix}\dfrac{\partial^2\Xi_i}{\partial z_{[i-1]}^2}z_i & 2\dfrac{\partial\Xi_i}{\partial z_{[i-1]}}\\ 2\left(\dfrac{\partial\Xi_i}{\partial z_{[i-1]}}\right)^\tau & 0\end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_i\end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_i\end{bmatrix}^\tau\right]z_i\\
&+\Xi_1\left[\varphi_1(0)+\overline{\Phi}_{11}(z_1)z_1\right]\left[\varphi_1(0)+\overline{\Phi}_{11}(z_1)z_1\right]^\tau\\
&+\textstyle\sum_{i=2}^{n}\Xi_i\left[\overline{\Phi}_i(0)+\sum_{j=1}^{i-1}\overline{\Phi}_{ij}(z_{[j]})z_j\right]\left[\overline{\Phi}_i(0)+\sum_{j=1}^{i-1}\overline{\Phi}_{ij}(z_{[j]})z_j\right]^\tau\\
\leqslant\;&\frac{1}{2}\sum_{i=1}^{n}\mathrm{tr}\left[\begin{bmatrix}\dfrac{\partial^2\Xi_i}{\partial z_{[i-1]}^2}z_i & 2\dfrac{\partial\Xi_i}{\partial z_{[i-1]}}\\ 2\left(\dfrac{\partial\Xi_i}{\partial z_{[i-1]}}\right)^\tau & 0\end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_i\end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_i\end{bmatrix}^\tau\right]z_i\\
&+\Xi_1\left[\varphi_1(0)+\overline{\Phi}_{11}(z_1)z_1\right]\left[\varphi_1(0)+\overline{\Phi}_{11}(z_1)z_1\right]^\tau\\
&+2\textstyle\sum_{i=2}^{n}\Xi_i\|\overline{\Phi}_i(0)\|^2+2\sum_{i=2}^{n}i\Xi_i\sum_{j=1}^{i-1}\Xi_j^{-1}\|\overline{\Phi}_{ij}(z_{[j]})\|^2\Xi_j z_j^2.
\end{aligned}\tag{23}
$$

Substituting (18)—(23) into (17) gives

$$
\begin{aligned}
\mathcal{L}V\leqslant\;&-\|\tilde{x}_{[r+1,n]}\|^2+\mathrm{tr}\left[\overline{H}(0)P\overline{H}^\tau(0)\right]+\mathrm{tr}\left\{\left[2\overline{H}(0)+\overline{H}_{11}(z_1)z_1\right]P\overline{H}_{11}^\tau(z_1)\right\}z_1\\
&+\varepsilon_1\|\tilde{x}_{[r+1,n]}\|^2+2\sum_{i=1}^{n}\Xi_i(z_{i+1}+\alpha_i+\Omega_i)z_i\\
&+\sum_{i=2}^{n}\sum_{j=1}^{i-1}\frac{\partial\Xi_i}{\partial z_j}(z_{j+1}+\alpha_j+\Omega_j)z_i^2\\
&+\frac{\|P\|^2}{\varepsilon_1}\left(\|G\|^2+1\right)\sum_{i=1}^{n}\left[\frac{\gamma_i}{\gamma}\delta\left(\|\chi\|\right)+z_1^2\delta_i(z_1)\right]+\frac{1}{2\varepsilon_2}\sum_{i=r}^{n}M_i^2z_i^2\\
&+\frac{\varepsilon_2}{2}|\tilde{x}_{r+1}|^2-\frac{\varepsilon_2}{2}\left|\tilde{x}_{r+1}-\frac{1}{\varepsilon_2}\sum_{i=r}^{n}M_iz_i\right|^2+\frac{1}{\varepsilon_2}\sum_{i=r}^{n}\sum_{j=r}^{i-1}M_jz_jM_iz_i\\
&+\sum_{i=1}^{r}\left[\Xi_i^2z_i^2+\frac{\gamma_i}{\gamma}\delta\left(\|\chi\|\right)+z_1^2\delta_i(z_1)\right]
\end{aligned}
$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{\min\{i-1,r\}}\left[\left(\frac{\partial\alpha_{i-1}}{\partial x_j}\right)^2\Xi_i^2 z_i^2+\frac{\gamma_j}{\gamma}\delta\left(\|\chi\|\right)+z_1^2\delta_j(z_1)\right]$$
$$+\frac{1}{2}\sum_{i=2}^{n}\left\{\sum_{j=1}^{\min\{i-1,r\}}\left[\left(\frac{\partial\Xi_i}{\partial z_j}\right)^2 z_i^4+\frac{\gamma_j}{\gamma}\delta\left(\|\chi\|\right)+z_1^2\delta_j(z_1)\right]\right.$$
$$\left.+\sum_{j=1}^{i-1}\sum_{k=1}^{\min\{j-1,r\}}\left[\left(\frac{\partial\Xi_i}{\partial z_j}\right)^2\left(\frac{\partial\alpha_{j-1}}{\partial x_k}\right)^2 z_i^4+\frac{\gamma_k}{\gamma}\delta\left(\|\chi\|\right)+z_1^2\delta_k(z_1)\right]\right\}$$
$$+\frac{1}{2}\sum_{i=1}^{n}\mathrm{tr}\left[\begin{bmatrix}\dfrac{\partial^2\Xi_i}{\partial z_{[i-1]}^2}z_i & 2\dfrac{\partial\Xi_i}{\partial z_{[i-1]}}\\ 2\left(\dfrac{\partial\Xi_i}{\partial z_{[i-1]}}\right)^\tau & 0\end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_i\end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_i\end{bmatrix}^\tau\right]z_i$$
$$+\Xi_1\left[2\varphi_1(0)+\overline{\Phi}_{11}(z_1)z_1\right]\overline{\Phi}_{11}^\tau(z_1)z_1$$
$$+\Xi_1\varphi_1(0)\varphi_1^\tau(0)+2\sum_{i=2}^{n}\Xi_i\|\overline{\Phi}_i(0)\|^2+2\sum_{i=2}^{n}i\Xi_i\sum_{j=1}^{i-1}\Xi_j^{-1}\|\overline{\Phi}_{ij}(z_{[j]})\|^2\Xi_j z_j^2. \tag{24}$$

Choose the weighted functions Ξ_1, $\Xi_i(z_{[i-1]})$ $(2\leqslant i\leqslant n)$ as

$$\Xi_1=\kappa_1,\quad \Xi_i=\frac{\kappa_i}{1+\|\overline{\Phi}_i(0)\|^2+\sum_{j=1}^{i-1}\Xi_j^{-1}\|\overline{\Phi}_{ij}(z_{[j]})\|^2},\qquad 2\leqslant i\leqslant n.$$

From (24) and the inequality:

$$2\sum_{i=1}^{n}\Xi_i\overline{\alpha}_i(0)z_i\sqrt{\frac{\Xi_i(0)}{\Xi_i(z_{[i-1]})}}\leqslant\sum_{i=1}^{n}\left[\frac{\Xi_i}{\varepsilon_{2i}}z_i^2+\Xi_i(0)\varepsilon_{2i}(\overline{\alpha}_i(0))^2\right]$$
$$\leqslant\sum_{i=1}^{n}\left[\frac{\Xi_i}{\varepsilon_{2i}}z_i^2+\kappa_i\varepsilon_{2i}(\overline{\alpha}_i(0))^2\right],$$

by straightforward calculations we arrive at

$$\mathcal{L}V\leqslant-\overline{c}_1\|\tilde{x}_{[r+1,n]}\|^2-\overline{c}_2\delta\left(\|\chi\|\right)-\sum_{i=1}^{n}\bar{\beta}_i\Xi_i z_i^2+\varepsilon_0\left[\gamma_0\delta\left(\|\chi\|\right)-z_1\delta_0(z_1)\right]$$
$$+2\sum_{i=1}^{n}\Xi_i\left[\alpha_i-\overline{\alpha}_i(X_{[i]})+\overline{\alpha}_i(0)\sqrt{\frac{\Xi_i(0)}{\Xi_i(z_{[i-1]})|_{z_{[i-1]}=\vartheta_{i-1}(X_{[i-1]})}}}\right]z_i+c_3, \tag{25}$$

where

$$\overline{c}_1=1-\varepsilon_1-\frac{\varepsilon_2}{2},$$
$$\overline{c}_2=\varepsilon_0\gamma_0-\sum_{i=1}^{r}\frac{\gamma_i}{\gamma}-\frac{\|P\|^2}{\varepsilon_1}\left(\|G\|^2+1\right)\sum_{i=1}^{n}\frac{\gamma_i}{\gamma}-\sum_{i=1}^{n}\sum_{j=1}^{\min\{i-1,r\}}\frac{\gamma_j}{\gamma}$$
$$-\frac{1}{2}\sum_{i=2}^{n}\left[\sum_{j=1}^{\min\{i-1,r\}}\frac{\gamma_j}{\gamma}+\sum_{j=1}^{i-1}\sum_{k=1}^{\min\{j-1,r\}}\frac{\gamma_k}{\gamma}\right],$$

$$\bar{\beta}_i = \beta_i - 2\sum_{j=i+1}^{n} j\kappa_j - \frac{1}{\varepsilon_{2i}}, \quad i = 1, \cdots, n-1, \quad \bar{\beta}_n = \beta_n - \frac{1}{\varepsilon_{2n}};$$

$$\overline{\alpha}_1 = \left\{ \frac{\varepsilon_0}{2\Xi_1}\delta_0(z_1) - \frac{\beta_1 z_1}{2} - \Omega_1 - \frac{1}{2\Xi_1}\mathrm{tr}\left\{\left[2\overline{H}(0) + \overline{H}_{11}(z_1)z_1\right] P\overline{H}_{11}^{\tau}(z_1)\right\} \right.$$

$$-\frac{1}{2}\Xi_1 z_1 - \frac{1}{2\Xi_1}\sum_{i=1}^{r}\delta_i(z_1)z_1 - \frac{1}{2}\left[2\varphi(0) + \overline{\Phi}_{11}(z_1)z_1\right]\overline{\Phi}_{11}^{\tau}(z_1)$$

$$-\frac{1}{2\Xi_1}\sum_{i=1}^{n}\sum_{j=1}^{\min\{i-1,r\}} z_1\delta_j(z_1)$$

$$\left. -\frac{z_1}{4\Xi_1}\sum_{i=2}^{n}\left[\sum_{j=1}^{\min\{i-1,r\}}\delta_j(z_1) + \sum_{j=1}^{i-1}\sum_{k=1}^{\min\{j-1,r\}}\delta_k(z_1)\right]\right\}_{z_1=x_1},$$

$$\overline{\alpha}_i = \left\{ -\frac{\beta_i z_i}{2} - \frac{\Xi_{i-1}}{2\Xi_i}z_{i-1} - \Omega_i - \frac{z_i}{2\Xi_i}\sum_{j=1}^{i-1}\frac{\partial \Xi_i}{\partial z_j}(z_{j+1} + \alpha_j + \Omega_j) \right.$$

$$-\frac{z_i}{2}\Xi_i - \frac{z_i}{2}\sum_{j=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_j}\right)^2 \Xi_i - \frac{1}{4\Xi_i}\sum_{j=1}^{i-1}\left(\frac{\partial \Xi_i}{\partial z_j}\right)^2 z_i^3$$

$$-\frac{1}{4\Xi_i}\mathrm{tr}\left[\begin{bmatrix} \dfrac{\partial^2 \Xi_i}{\partial z_{[i-1]}^2}z_i & 2\dfrac{\partial \Xi_i}{\partial z_{[i-1]}} \\ 2\left(\dfrac{\partial \Xi_i}{\partial z_{[i-1]}}\right)^{\tau} & 0 \end{bmatrix}\begin{bmatrix}\Phi_1 \\ \vdots \\ \Phi_i\end{bmatrix}\begin{bmatrix}\Phi_1 \\ \vdots \\ \Phi_i\end{bmatrix}^{\tau}\right]$$

$$\left. -\frac{1}{4\Xi_i}\sum_{j=1}^{i-1}\sum_{k=1}^{j-1}\left(\frac{\partial \Xi_i}{\partial z_j}\right)^2\left(\frac{\partial \alpha_{j-1}}{\partial x_k}\right)^2 z_i^3\right\}_{z_{[i]}=\vartheta_i(X_{[i]})}, i = 2, \cdots, r-1,$$

$$\overline{\alpha}_r = \left\{ -\frac{\beta_r z_r}{2} - \frac{\Xi_{r-1}}{2\Xi_r}z_{r-1} - \Omega_r - \frac{z_r}{2\Xi_r}\sum_{j=1}^{r-1}\frac{\partial \Xi_r}{\partial z_j}(z_{j+1} + \alpha_j + \Omega_j) \right.$$

$$-\frac{z_r}{2}\Xi_r - \frac{z_r}{2}\sum_{j=1}^{r-1}\left(\frac{\partial \alpha_{r-1}}{\partial x_j}\right)^2 \Xi_r - \frac{1}{4\Xi_r}\sum_{j=1}^{r-1}\left(\frac{\partial \Xi_r}{\partial z_j}\right)^2 z_r^3$$

$$-\frac{1}{4\Xi_r}\mathrm{tr}\left[\begin{bmatrix} \dfrac{\partial^2 \Xi_r}{\partial z_{[r-1]}^2}z_r & 2\dfrac{\partial \Xi_r}{\partial z_{[r-1]}} \\ 2\left(\dfrac{\partial \Xi_r}{\partial z_{[r-1]}}\right)^{\tau} & 0 \end{bmatrix}\begin{bmatrix}\Phi_1 \\ \vdots \\ \Phi_r\end{bmatrix}\begin{bmatrix}\Phi_1 \\ \vdots \\ \Phi_r\end{bmatrix}^{\tau}\right]$$

$$\left. -\frac{1}{4\Xi_r\varepsilon_2}M_r^2 z_r - \frac{1}{4\Xi_r}\sum_{j=1}^{r-1}\sum_{k=1}^{j-1}\left(\frac{\partial \Xi_r}{\partial z_j}\right)^2\left(\frac{\partial \alpha_{j-1}}{\partial x_k}\right)^2 z_r^3\right\}_{z_{[r]}=\vartheta_r(X_{[r]})},$$

$$\overline{\alpha}_i = \Bigg\{ -\frac{\beta_i z_i}{2} - \frac{\Xi_{i-1}}{2\Xi_i} z_{i-1} - \Omega_i - \frac{z_i}{2\Xi_i}\sum_{j=1}^{i-1}\frac{\partial \Xi_i}{\partial z_j}(z_{j+1}+\alpha_j+\Omega_j)$$

$$-\frac{z_i}{2}\sum_{j=1}^{r}\left(\frac{\partial \alpha_{i-1}}{\partial x_j}\right)^2 \Xi_i - \frac{1}{4\Xi_i\varepsilon_2}M_i^2 z_i - \frac{M_i}{2\Xi_i\varepsilon_2}\sum_{j=r}^{i-1}M_j z_j$$

$$-\frac{1}{4\Xi_i}\mathrm{tr}\left[\begin{bmatrix} \dfrac{\partial^2 \Xi_i}{\partial z_{[i-1]}^2} z_i & 2\dfrac{\partial \Xi_i}{\partial z_{[i-1]}} \\ 2\left(\dfrac{\partial \Xi_i}{\partial z_{[i-1]}}\right)^{\tau} & 0 \end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots \\ \Phi_i\end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots \\ \Phi_i\end{bmatrix}^{\tau}\right] - \frac{1}{4\Xi_i}\sum_{j=1}^{r}\left(\frac{\partial \Xi_i}{\partial z_j}\right)^2 z_i^3$$

$$-\frac{1}{4\Xi_i}\sum_{j=1}^{i-1}\left(\frac{\partial \Xi_i}{\partial z_j}\right)^2 \sum_{k=1}^{\min\{j-1,r\}}\left(\frac{\partial \alpha_{j-1}}{\partial x_k}\right)^2 z_i^3\Bigg\}_{z_{[i]}=\vartheta_i(X_{[i]})}, \; i = r+1, \cdots, n;$$

$$c_3 = \mathrm{tr}\left[\overline{H}(0)P\overline{H}^{\tau}(0)\right] + \kappa_1\varphi_1(0)\varphi_1^{\tau}(0) + \textstyle\sum_{i=2}^{n}\kappa_i\left(\dfrac{2\|\overline{\Phi}(0)\|^2}{1+\|\overline{\Phi}(0)\|^2} + \varepsilon_{2i}\overline{\alpha}_i^2(0)\right). \quad (26)$$

Design $\alpha_i\ (i=1,\cdots,n)$ as follows

$$\alpha_i(X_{[i]}) = \overline{\alpha}_i(X_{[i]}) - \overline{\alpha}_i(0)\sqrt{\frac{\Xi_i(0)}{\Xi_i(z_{[i-1]})|_{z_{[i-1]}=\vartheta_{i-1}(X_{[i-1]})}}}. \quad (27)$$

Then, the actual control is

$$u = \alpha_n. \quad (28)$$

By substituting (27) and (28) into (25) we have

$$\mathcal{L}V \leqslant -\overline{c}_1\|\tilde{x}_{[r+1,n]}\|^2 - \overline{c}_2\delta(\|\chi\|) - \sum_{i=1}^{n}\bar{\beta}_i\Xi_i z_i^2 + \varepsilon_0[\gamma_0\delta(\|\chi\|) - z_1\delta_0(z_1)] + c_3. \quad (29)$$

3.3 Choice of design parameters

From the above design procedure we see that the key point is to choose the positive design parameters ε_0, ε_1, ε_2 and κ_i, ε_{2i}, β_i $(i=1,\cdots,n)$ such that

$$\overline{c}_1 > 0, \quad \overline{c}_2 \geqslant 0, \quad \bar{\beta}_1 > 0, \cdots, \bar{\beta}_n > 0. \quad (30)$$

The following lemma tells us a range and method specifying these design parameters.

Lemma 1. There are always positive design parameters ε_0, ε_1, ε_2 and κ_i, ε_{2i}, β_i $(i=1,\cdots,n)$ such that (30) holds.

Proof. Choose arbitrarily design parameters $\kappa_i > 0$, $\varepsilon_{2i} > 0$ $(i=1,\cdots,n)$, $\varepsilon_1 \in (1, \frac{1}{2})$, $\varepsilon_2 \in (0,1)$, and choose

$$\beta_i > 2\sum_{j=i+1}^{n} j\kappa_j + \frac{1}{\varepsilon_{2i}}(i=1,\cdots,n-1); \quad \beta_n > \frac{1}{\varepsilon_{2n}}$$

and

$$\varepsilon_0 \geqslant \sum_{i=1}^{r} \frac{\gamma_i}{\gamma_0 \gamma} + \sum_{i=1}^{n} \frac{\|P\|^2}{\gamma_0 \gamma \varepsilon_1} \left(\|G\|^2 + 1\right) \gamma_i + \frac{1}{\gamma_0 \gamma} \sum_{i=1}^{n} \sum_{j=1}^{\min\{i-1,\, r\}} \gamma_j + \frac{1}{2\gamma_0 \gamma} \sum_{i=2}^{n} \left[\sum_{j=1}^{\min\{i-1,\, r\}} \gamma_j + \sum_{j=1}^{i-1} \sum_{k=1}^{\min\{j-1,\, r\}} \gamma_k \right].$$

Then, it can easily be seen that (30) holds.

3.4 Main results

The following theorem summarizes the main results of this paper.

Theorem 2. Consider the nonlinear stochastic system (2). Suppose Assumptions A1—A4 hold, the design parameters $\varepsilon_0, \varepsilon_1, \varepsilon_2$ and $\kappa_i, \varepsilon_{2i}, \beta_i$ $(i = 1, \cdots, n)$ are chosen to satisfy (30). Then the minimal-order observer (7) based output-feedback control (28) is such that the closed-loop system has a unique solution on $[0, \infty)$ almost surely, and is bounded in probability. Furthermore, when $\varphi_i(0) = 0 (i = 1, \cdots, n)$, the closed-loop system is asymptotically stable in the large.

Proof. We have completed the control design in subsection 3.2. Suppose the Lyapunov function for the whole system is $V_c = \varepsilon_0 V_0 + V$. Then, by Assumption A1 and (29) we have

$$\mathcal{L}V_c \leqslant -\varepsilon_0 c V_0 - \overline{c}_1 \|\tilde{x}_{[r+1,n]}\|^2 - \overline{c}_2 \delta\left(\|\chi\|\right) - \sum_{i=1}^{n} \bar{\beta}_i \Xi_i z_i^2 + c_3 \leqslant -c_1 V_c + c_3, \quad (31)$$

where $c_1 = \min\left(c, \overline{c}_1 \lambda_{\max}^{-1}(P), \bar{\beta}_1, \cdots, \bar{\beta}_n\right)$. Again, by Assumption A1 we get $\varepsilon_0 W_{01} + V \leqslant V_c \leqslant \varepsilon_0 W_{02} + V$, where $\varepsilon_0 W_{01} + V$ and $\varepsilon_0 W_{02} + V$ are positive, radially unbounded. Then, from Theorem 1 it follows that the closed-loop system has a unique solution on $[0, \infty)$ almost surely, and is bounded in probability.

If $\varphi_i(0) = 0\,(i = 1, \cdots, n)$, then we have $\overline{H}(0) = 0$, $\overline{\Phi}_i(0) = 0$ and $\overline{\alpha}_i(0) = 0$. This leads to $c_3 = 0$ and $\mathcal{L}V_c \leqslant -c_1 V_c$, which together with Theorem 1 implies that the closed-loop system is asymptotically stable in the large.

Remark 3. If the observer (13) is adopted for control design, then an extra term $2\tilde{x}_{[r+1,\,n]}^{\tau} P \overline{f}(y)$ will arise on the right-hand side of (17). To deal with this term, we first separate $\tilde{x}_{[r+1,\,n]}$ and $\overline{f}(y)$ by using the Young's inequality:

$$2\tilde{x}_{[r+1,\,n]}^{\tau} P \overline{f}(y) \leqslant \varepsilon_3 \|\tilde{x}_{[r+1,\,n]}\|^2 + \frac{\|P\|^2}{\varepsilon_3} \|\overline{f}(y)\|^2, \quad \varepsilon_3 > 0,$$

where $\varepsilon_3 \|\tilde{x}_{[r+1,\,n]}\|^2$ can be dominated by the negative terms on the right-hand side of (17). While for term $\frac{\|P\|^2}{\varepsilon_3} \|\overline{f}(y)\|^2$, in the case where $r = 1$, thanks to the existence of the factor z_1^2, it can be cancelled out by the virtual control α_1 (see ref. [10]), but when $r > 1$, $\frac{\|P\|^2}{\varepsilon_3} \|\overline{f}(y)\|^2$ cannot be controlled effectively by the virtual control α_i $(i = 1, \cdots, r-1)$, since there is no further assumption on $\overline{f}(y)$, and by integrator backstepping approach, measurable states $x_{i+1}, \cdots, x_r$ are unavailable for virtual controls α_i $(i = 1, \cdots, r-1)$.

And, $\frac{\|P\|^2}{\varepsilon_3}\|\overline{f}(y)\|^2$ cannot be cancelled out by the virtual controls α_i $(i = r, \cdots, n-1)$ and the actual control u, since z_i $(i = r, \cdots, n)$ is not a factor of $\frac{\|P\|^2}{\varepsilon_3}\|\overline{f}(y)\|^2$. Thus, it is hard to control effectively the term $2\tilde{x}^{\tau}_{[r+1,\, n]}P\overline{f}(y)$. Therefore, the observer (13) is not adequate to the multi-output ($y = x_{[r]}, r > 1$) systems.

Remark 4. From (27) and (28) it follows that the virtual controls α_i $(i = 1, \cdots, n-1)$ and the actual control $u = \alpha_n$ preserve the equilibrium at the origin of the nonlinear system. c_3 depicts the static property of the closed-loop system. From (26), we can see that the smaller the design parameters κ_i and ε_{2i} $(i = 1, \cdots, n)$ are, the smaller c_3 is. This together with (31) implies that in order to get a small static upper bound of the closed-loop system states, we only need to take small κ_i and ε_{2i} $(i = 1, \cdots, n)$. However, from the expressions of virtual controls and actual control, we see that the smaller the design parameters κ_i and ε_{2i} $(i = 1, \cdots, n)$ are, the more the control energy needs, in other words, in order to get a small static upper bound of the closed-loop system state, one should pay more in terms of control energy.

4 Conclusion

In this paper, the design problem of output-feedback stabilization control for a class of SIMO stochastic nonlinear systems with unobservable states, unmodelled dynamics and stochastic disturbances is investigated, and the design methods of minimal-order observer and output-feedback stabilization control are presented. Based on the observer designed, the estimates of all the system states are given, and the convergence of the estimation error is analyzed. By using integrator backstepping approach, an output-feedback stabilization controller is constructively designed, which ensures that the closed-loop system is bounded in probability and, when the nonlinearities and stochastic disturbance equal zero, asymptotically stable in the large. The observer introduced in this paper not only preserves the advantages of the full-order observer, but also avoids the extra term arising in the dynamical equation of the estimation error, and so, is adequate to the control design of SIMO systems (e.g. $y = (x_1, \cdots, x_r)^{\tau}$, $r > 1$). Problem needing further study includes: how to constructively design minimal-order and the observer-based output-feedback stabilization control for systems, such as, with output of the form $y = Cx$, where C is an $r \times n$ matrix with only one element 1 in each row and column and the others are 0.

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References

1. Deng, H., Krstić, M., Output-feedback stochastic nonlinear stabilization, IEEE Transactions on Automatic Control, 1999, 44: 328—333.
2. Deng, H., Krstić, M., Williams, R., Stabilization of stochastic nonlinear systems driven by noise of unknown covariance, IEEE Transactions on Automatic Control, 2001, 46: 1237—1253.
3. Florchinger, P., Lyapunov-like techniques for stochastic stability, SIAM J. Control and Optimization, 1995, 33: 1151—1169.

4. Krstić, M., Deng, H., Stabilization of Nonlinear Uncertain Systems, London: Springer-Verlag, 1998.
5. Liu, Y. G., Pan, Z., Shi, S. J., Output feedback control design for strict feedback stochastic nonlinear systems under a risk-sensitive cost criterion, Proceedings of the 40th IEEE Conference on Decision and Control (Florida), Piscataway: IEEE Control Systems Society, 2001, 1269—1274.
6. Liu, Y. G., Zhang, J. F., Pan, Z. G., Design of satisfaction output feedback controls for stochastic nonlinear systems under quadratic tracking risk-sensitive index, Science in China (Series F), 2003, 46(2): 126—144.
7. Pan, Z. G., Basar, T., Backstepping controller design for nonlinear stochastic systems under a risk-sensitive cost criterion, SIAM J. Control and Optimization, 1999, 37: 957—995.
8. Pan, Z. G., Liu, Y. G., Shi, S. J., Output feedback stabilization for stochastic nonlinear systems in observer canonical form with stable zero-dynamics, Science in China (Series F), 2001, 44(4): 292—308.
9. Khas'minskii, R. Z., Stochastic Stability of Differential Equations, Rockville: S and N International Publishers, 1980.
10. Jiang, Z. P., Global output feedback control with disturbance attenuation for minimum-phase nonlinear systems, Systems and Control Letters, 2000, 39: 155—164.
11. Gihman, R. A., Skorohod, A. V., Stochastic Differential Equations, New York: Springer-Verlag, 1972.
12. Wang, E. P., Qin, H. S., Wang, S. L., Introduction to Linear Control System Theory, Guangzhou: Guangdong Science and Technology Press, 1991.